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HW 4

MATH 563
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1. Koralov and Sinai, Ch. 2 Problem 5

   Using the de Moivre-Laplace Theorem, estimate the probability that during 12000 cosses of a die the number 6 appeared between 1900 and 2100 times.

   Solution. By the de Moivre-Laplace Theorem, we can make the approximation

$$\binom{n}{k} p^k(1-p)^{n-k} \approx \frac{1}{\sqrt{2\pi np(1-p)}} e^{\frac{-(k-np)^2}{2npq}}.$$

   For this problem, $n = 12000$ $p = \frac{1}{6}$ and $1 - p = \frac{5}{6}$. The sum over all outcomes of interest can then be approximated by the integral

$$\mathrm{P}(1900 \leq k \leq 2100) \approx \frac{1}{\sqrt{\pi}} \int_{d_1}^{d_2} e^{-u^2} du$$

   using the appropriate change of variables. The limits of integration are

$$\begin{aligned} d_1 &= \frac{1900 - np}{\sqrt{2np(1-p)}} \\ d_2 &= \frac{2100 - np}{\sqrt{2np(1-p)}}. \end{aligned}$$

   The particular change of variables was selected to use erf function output from Matlab. The integral evaluates to

$$\begin{aligned} \mathrm{P}(1900 \leq k \leq 2100) &\approx \frac{1}{\sqrt{\pi}} \int_{d_1}^{d_2} e^{-u^2} du \\ &\approx 0.9857 \end{aligned}$$

2. *Koralov and Sinai, Ch. 2 Problem 7

   Two candidates were running for a post. One received 520000 votes and the other 480000 votes. Afterwards it became apparent that the voting machines were defective- they randomly and independently switched each vote for the opposite one with probability of 45 percent. The losing candidate asked for a re-vote. Is there a basis for a re-vote?

   Solution. I would like to find a number of votes v_1 s.t. the probability that more than v_1 votes were flipped is around 0.1. This number was subjectively picked, however the process that follows the selection shows that there is no basis for a re-vote. Using the de Moivre-Laplace Theorem, I find that there is a 0.0784 chance that greater than 234508 votes were flipped out of the 520000 that the first candidate received since

$$\mathrm{P}(k \geq 234508) \approx \frac{1}{\sqrt{\pi}} \int_{d_1}^{d_2} e^{-u^2} du \approx 0.0784$$

where

$$\begin{aligned} d_1 &= \frac{234508 - np}{\sqrt{2np(1-p)}} \\ d_2 &= \frac{520000 - np}{\sqrt{2np(1-p)}} \\ n &= 520000 \\ p &= 0.45 \\ 1-p &= 0.55. \end{aligned}$$

In order for there to be a basis for a recount, there will have to be non-trivial probability that less than 214508 votes were flipped from the 480000 votes for candidate 2. Following the same process as above, the probability is 7.45×10^{-6}. Since the vote switching is independent, the total probability of interest is (where A is the event that > 234508 votes were switched from candidate 1 and B is the event that < 214508 votes were switched from candidate 2).

$$\begin{aligned} \mathrm{P}(A \cap B) &= \mathrm{P}(A)\mathrm{P}(B) \\ &\approx 5.96 \times 10^{-7} \end{aligned}$$

which is much too small of a chance to justify a re-vote.

3. *Koralov and Sinai, Ch. 2 Problem 9

   Suppose that during one day the price of a certain stock either goes up by 3 percent with probability 1/2 or goes down by 3 percent with probability 1/2, and that outcomes on different days are independent. Approximate the probability that after 250 days the price of the stock will be at least as high as the current price.

   Solution. Consider the quantity

$$V = (1.03)^k (0.97)^{250-k} V_0$$

The value V of the stock will be greater than the initial value V_0 if the quantity $(1.03)^k(0.97)^{250-k}$ is greater than 1. This occurs for $k \geq 127$. The approximation of the probability that after 250 days the stock maintains at least its original price is

$$\mathrm{P}(k \geq 127) \approx \int_{d_1}^{d_2} e^{-u^2} du \approx 0.4001$$

where

$$\begin{aligned} d_1 &= \frac{127 - np}{\sqrt{2np(1-p)}} \\ d_2 &= \frac{250 - np}{\sqrt{2np(1-p)}} \\ n &= 250 \\ p &= 0.5 \\ 1-p &= 0.5. \end{aligned}$$

4. Koralov and Sinai, Ch. 4 Problem 1

Let P be the probability distribution of the sequence of n Bernoulli trials, $\omega = (\omega_1, \ldots, \omega_n)$, $\omega_i = 1$ or 0 with probabilities p and $1-p$. Find $\mathrm{P}(\omega_1 = 1 | \omega_1 + \cdots + \omega_n = m)$.

Solution. Consider A, the event that $\omega_1 = 1$, and B, the event that $\sum_{i=1}^n \omega_i = m$.

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

$\mathrm{P}(B)$ represents the probability that there will be m successes ($\omega_i = 1$) in n trials, that is

$$\mathrm{P}(B) = \binom{n}{k} p^m (1-p)^{n-m}$$

Now, the probability of $A \cap B$ represents the probability of the event that $\omega_1 = 1$ and there are $m-1$ successes in the remaining $n-1$ trials, that is

$$\mathrm{P}(A \cap B) = p\binom{n-1}{m-1} p^{m-1}(1-p)^{n-m}$$

Then,

$$\begin{aligned}\mathrm{P}(A|B) &= \frac{\binom{n-1}{m-1}}{\binom{n}{m}} \\ &= \frac{m}{n}\end{aligned}$$

5. Koralov and Sinai, Ch. 4 Problem 2

Find the distribution function of a random variable ξ which takes positive values and satisfies $\mathrm{P}(\xi > x + y | \xi > x) = \mathrm{P}(\xi > y)$ for all $x, y > 0$.

Solution. Assume WLOG $x = y$. In this case, the problem reduces to

$$\mathrm{P}(\xi > 2x | \xi > x) = \frac{\mathrm{P}(\xi > 2x)}{\mathrm{P}(\xi > x)} = \mathrm{P}(\xi > x)$$

This implies that

$$\mathrm{P}(\xi > 2x) = \mathrm{P}(\xi > x)^2.$$

Use the above relation with $y = (n-1)x$ to get

$$\mathrm{P}(\xi > nx) = \mathrm{P}(\xi > x)^n.$$

Look at the complement of this statement to get the problem in terms of distribution function

$$\begin{aligned} 1 - \mathrm{P}(\xi \le nx) &= (1 - \mathrm{P}(\xi \le x))^n \\ \implies 1 - (1 - \mathrm{P}(\xi \le nx))^{1/n} &= \mathrm{P}(\xi \le x) \\ \implies 1 - (1 - F(nx))^{1/n} &= F(x) \end{aligned}$$

By observation, $F(x) = 1 - e^{-x}$ satisfies this relation. Notice: ξ can only take positive values. This completes this problem.

6. Koralov and Sinai, Ch. 4 Problem 3

   Two coins are in a bag. One is symmetric, while the other is not - if tossed it lands heads up with probability equal to 0.6. One coin is randomly pulled out of the bag and tossed. It lands heads up. What is the probability that the same coin will land heads up if tossed again?

   Solution. Let A be the event that the fair coin was chosen and B be the event that heads was flipped on the first toss. Then

$$\begin{aligned} \mathrm{P}(A|B) &= \frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} \\ &= \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}0.6} \\ &= \frac{5}{11} \end{aligned}$$

   Therefore, the probability that the non-symmetric coin is picked is $\frac{6}{11}$. Now, the probability that heads is flipped given that heads was flipped first is

$$\begin{aligned} \mathrm{P}(\text{Heads}|\text{Heads}) &= \frac{5}{11}(0.5)+\frac{6}{11}(0.6) \\ &\approx 0.5545 \end{aligned}$$

7. *Koralov and Sinai, Ch. 4 Problem 4

   Suppose that each of the random variables ξ and η takes at most two values, a and b. Prove that ξ and η are independent if $\mathrm{E}(\xi\eta)=\mathrm{E}\xi\mathrm{E}\eta$.

   Solution.

8. *Koralov and Sinai, Ch. 4 Problem 5

   Give an example of three events A_1, A_2, and A_3 which are not independent, yet pairwise independent.

   Solution. Consider a die with sides colored red, green and blue. Let A_1 be the event that a red is rolled, A_2 be the event that a green is rolled, and A_3 be the event that a blue is rolled. Then the probability that any color is rolled is $\frac{1}{2}$.

$$\begin{aligned} \mathrm{P}(A_1\cap A_2)=\mathrm{P}(A_2\cap A_3)=\mathrm{P}(A_1\cap A_3) &= \frac{1}{4} \\ &= \mathrm{P}(A_1)\mathrm{P}(A_2)=\mathrm{P}(A_2)\mathrm{P}(A_3)=\mathrm{P}(A_1)\mathrm{P}(A_3). \end{aligned}$$

   Therefore, the events are pairwise independent. However,

$$\mathrm{P}(A_1\cap A_2\cap A_3)=\frac{1}{4}\neq\frac{1}{8}=\mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)$$

   so the events are not totally independent.

9. *Koralov and Sinai, Ch. 4 Problem 6

   Give an example of two random variables ξ and η which are not independent, yet $\mathrm{E}(\xi\eta) = \mathrm{E}\xi\mathrm{E}\eta$.

   Solution. Let ξ be uniformly distributed on $(-a, a)$ and $\eta = \xi^2$. Clearly, ξ and η are not independent.

$$\begin{aligned}
\mathrm{E}(\xi\eta) &= \mathrm{E}(\xi^3) \\
&= 0 \\
&= 0\mathrm{E}(\xi^2) \\
&= 0\mathrm{E}(\eta) \\
&= \mathrm{E}(\xi)\mathrm{E}(\eta).
\end{aligned}$$

   So, the events are not independent, but $\mathrm{E}(\xi\eta) = \mathrm{E}\xi\mathrm{E}\eta$.

10. *Koralov and Sinai, Ch. 4 Problem 7

   A random variable ξ has Gaussian distribution with mean zero and variance one, while a random variable η has the distribution with density

$$p_\eta(t) = \begin{cases} te^{-\frac{t^2}{2}} & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the distribution of $\zeta = \xi \cdot \eta$ assuming that ξ and η are independent.

   Solution.

11. Koralov and Sinai, Ch. 4 Problem 8

   Let ξ_1 and ξ_2 be two independent random variables with Gaussian distribution with mean zero and variance one. Prove that $\eta_1 = \xi_1^2 + \xi_2^2$ and $\eta_2 = \xi_1/\xi_2$ are independent.

   Proof. First, identify the density functions for η_1 and η_2. For η_1, the density is the χ^2 density with 2 degrees of freedom.

$$p_{\eta_1}(x_1) = \frac{1}{2\Gamma(1)}e^{-\frac{x_1}{2}}$$

   For η_2, Hahn and Shapiro gives the density as

$$p_{\eta_2}(x_2) = \frac{1}{2\pi}\int_\Omega \frac{\tau}{x_2^2}e^{-\frac{\tau^2}{2}}e^{-\frac{\tau^2}{2x_2^2}}d\tau.$$

   Consider the probability that $\eta_1 \in \eta_1(A)$ and $\eta_2 \in \eta_2(B)$.

$$\begin{aligned}
\mathrm{P}(\eta_1 \in \eta_1(A) \cap \eta_2 \in \eta_2(B)) &= \int_{x_2 \in B}\int_{x_1 \in A} p_{\eta_1}p_{\eta_2}dx_1dx_2 \\
&= \int_{x_2 \in B} p_{\eta_2}dx_2 \int_{x_1 \in A} p_{\eta_1}dx_1 \\
&= \mathrm{P}(x_1 \in A)\mathrm{P}(x_2 \in B)
\end{aligned}$$

   Therefore, η_1 and η_2 are independent. □

12. *Koralov and Sinai, Ch. 4 Problem 9

    Two editors were independently proof-reading the same manuscript. One found a misprints, the other found b misprints. Of those, c misprints were found by both of them. How would you estimate the total number of misprints in the manuscript?

    Solution.

13. Koralov and Sinai, Ch. 4 Problem 10

    Let ξ, η be independent Poisson distributed random variables with expectations λ_1 and λ_2 respectively. Find the distribution of $\zeta = \xi + \eta$.

    Solution. Suppose $\xi = k$, then $\zeta = z$ iff $\eta = z - k$. So, $\zeta = z$ is the union of pairwise disjoint events $A = \{\xi = k\}$ and $B = \{\eta = z - k\}$. Since the events are pairwise disjoint, the probability of the union over all possible values is the sum of the probabilities, that is

$$\begin{aligned} \mathrm{P}(\zeta = z) &= \sum_{k=0}^{\infty} \mathrm{P}(\xi = k)\mathrm{P}(\eta = z - k) \\ &= \sum_{k=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^k}{k!} \frac{e^{-\lambda_2}\lambda_2^{z-k}}{(z-k)!}. \end{aligned}$$

    Now, the distribution function represents the probability $\mathrm{P}(\zeta \leq z)$. Therefore,

$$\begin{aligned} F(z) &= \mathrm{P}(\zeta \leq z) \\ &= \sum_{n=0}^{z} \sum_{k=0}^{n} \frac{e^{-\lambda_1}\lambda_1^k}{k!} \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}. \end{aligned}$$

14. Koralov and Sinai, Ch. 4 Problem 11

    Let ξ, η be independent random variables. Assume that ξ has the uniform distribution on $[0, 1]$, and η has the Poisson distribution with parameter λ. Find the distribution of $\zeta = \xi + \eta$.

    Solution. Follow much the same process as in Problem 10. Suppose $\xi = x$ and $\zeta = z$. Then $\eta = z - x$. Then the probability that $\mathrm{P}(\zeta \leq z)$ is

$$F(z) = \mathrm{P}(\zeta \leq z) = \int_0^x \sum_{k=0}^{z-x} \frac{e^{-\lambda}\lambda^k}{k!} dt.$$

15. *Koralov and Sinai, Ch. 4 Problem 12

    Let $\xi_1, \xi_2, \ldots$ be independent identically distributed Gaussian random variables with mean zero and variance one. Let $\eta_1, \eta_2, \ldots$ be independent identically distributed exponential random variables with mean one. Prove that there is $n > 0$ such that

$$\mathrm{P}(\max(\eta_1, \ldots, \eta_n)) \geq \max(\xi_1, \ldots, \xi_n)) > 0.99.$$

Solution.

16. *Koralov and Sinai, Ch. 4 Problem 13

    Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent algebras, that is any two sets $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Prove that the σ-algebras $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are also independent. (Hint: use Lemma 4.12.)

    Proof. □

17. Koralov and Sinai, Ch. 4 Problem 14

    Let $\xi_1, \xi_2, \ldots$ be independent identically distributed random variables and N be an $\mathbb{N}$-valued random variable independent of ξ_i's. Show that if ξ_1 and N have finite expectation, then

$$\mathrm{E}\sum_{i=1}^{N} \xi_i = \mathrm{E}(N)\mathrm{E}(\xi_1)$$

Solution. Assume $\mathrm{E}(N), \mathrm{E}(\xi_1)$ are finite, then

$$\begin{aligned}
\mathrm{E}\sum_{i=1}^{N} \xi_i &= \mathrm{E}\left[\mathrm{E}\sum_{i=1}^{N} \xi_i\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{N} \mathrm{E}(\xi_i)\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{N} \mathrm{E}(\xi_1)\right] \text{ since RVs are identically distributed} \\
&= \mathrm{E}(\xi_1)\mathrm{E}\left[\sum_{i=1}^{N} 1\right] \\
&= \mathrm{E}(N)\mathrm{E}(\xi_1)
\end{aligned}$$